Published by Institute of Physics Publishing for SISSA

Received: *March 31, 2008*
Accepted: *May 19, 2008*
Published: *June 4, 2008*

Holomorphic maps and the complete $\frac{1}{N}$ expansion of 2D $\mathrm{SU}(N)$ Yang-Mills

Yusuke Kimura and Sanjaye Ramgoolam

Centre for Research in String Theory, Department of Physics,
Queen Mary University of London,
Mile End Road London E1 4NS U.K.
E-mail: y.kimura@qmul.ac.uk, s.ramgoolam@qmul.ac.uk

Abstract: We give a description of the complete $1/N$ expansion of $\mathrm{SU}(N)$ 2D Yang Mills theory in terms of the moduli space of holomorphic maps from non-singular worldsheets. This is related to the Gross-Taylor coupled $1/N$ expansion through a map from Brauer algebras to symmetric groups. These results point to an equality between Euler characters of moduli spaces of holomorphic maps from non-singular worldsheets with a target Riemann surface equipped with markings on the one hand and Euler characters of another moduli space involving worldsheets with double points (nodes).

Keywords: AdS-CFT Correspondence, Topological Strings, Matrix Models.

JHEP06(2008)015

Contents

1. Introduction

The exact partition function [1] of two dimensional Yang Mills (2DYM) for SU(N) gauge group on a Riemann surface of genus G and area A is given by

$$Z_{G,A} = \sum_R (DimR)^{2-2G} e^{-g^2_{\rm YM} A C_2(R)} \tag{1.1}$$

Of special interest to us is the zero area limit

$$Z_G = \sum_R (DimR)^{2-2G} \tag{1.2}$$

R runs over irreducible representations of SU(N) and $Dim(R)$ is the dimension of the representation. The results for manifolds with boundary are also known. Our main interest will be in the situation where $\chi = 2-2G-B \leq -1$. The string theory interpretation of the $\frac{1}{N}$ expansion was developed in [2]. For earlier work on stringy aspects of 2DYM, see [3]. For a review of the exact partition function, its $\frac{1}{N}$ expansion, and the string theory interpretation see [4]. $\Sigma(G,B)$ in this paper denotes a Riemann surface of genus G with B boundaries.

The large N expansion of these partition functions is described in terms of a coupling of a chiral partition function Z^+ with an anti-chiral partition function Z^- [2]. The chiral parition function is obtained by replacing $\sum_R \rightarrow \sum_n \sum_{R \vdash n}$

$$Z_G^+ = \sum_{n=0}^{\infty} \sum_{R \vdash n} (DimR)^{2-2G} \tag{1.3}$$

R now runs over Young diagrams with n boxes. Using Schur-Weyl duality, which relates the actions of U(N) (or SU(N)) and S_n in $V^{\otimes n}$, this can be manipulated to give

$$Z_G^+ = \sum_n \frac{N^{n(2-2G)}}{n!} \sum_{s_1,t_1 \cdots s_G,t_G \in S_n} \delta_n \left(\Omega_n^{2-2G} \prod_{i=1}^{G} s_i t_i s_i^{-1} t_i^{-1} \right) \tag{1.4}$$

where the "chiral Ω factor" Ω_n is an element in the group algebra $\mathbb{C}(S_n)$

$$\Omega_n = \sum_{\sigma \in S_n} N^{C_\sigma - n}\ \sigma \tag{1.5}$$

The δ_n is defined over S_n by

$$\delta_n(\sigma) = 1 \text{ if } \sigma = 1 \text{ and } 0 \text{ otherwise} \tag{1.6}$$

and extended over the group algebra $\mathbb{C}(S_n)$ by linearity. The expansion (1.4) can be used to show that each order in the $(\frac{1}{N})^{2g-2}$ expansion of Z^+ is a sum over equivalence classes of branched covers from a worldsheet Riemann surface of genus g to the target Σ_G, so that we have a topological string theory with $g_s = \frac{1}{N}$. It is useful to define $b(\sigma) = n - C_\sigma$ which is the branching number of the permutation. The Riemann-Hurwitz formula

$$2g - 2 = n(2G-2) + \sum_i b(\sigma_i) \tag{1.7}$$

gives the Euler character of the worldsheet for a branched cover of Σ_G with branchings i described by σ_i. So we see that the power $N^{-b(\sigma)}$ appearing in the Ω factor is compatible with the interpretation of Ω_n in terms of branch points with $g_s = \frac{1}{N}$. The Ω_n can be written as $1 + \sum' \sigma N^{-b(\sigma)}$ and we have the expansion

$$\Omega_n^{2-2G} = \sum_{L=0}^{\infty} d(2-2G, L) \sum_{\sigma_1, \ldots \sigma_L}^{\prime} \sigma_1 \sigma_2 \cdots \sigma_L\ N^{-b(\sigma_1) - b(\sigma_2) \cdots\ -\ b(\sigma_L)} \tag{1.8}$$

where $d(2-2G, L)$ is a binomial coefficient. The $L = 0$ term is defined as 1. This factor $d(2-2G, L)$, related to the exponent $2-2G$ in Ω_n^{2-2G}, is the Euler character of the configuration space of L indistiguishable points on Σ_G. Along with the structure of Hurwitz spaces as discrete fibrations over these configuration spaces, (1.4) and (1.8) are used to prove that the Z_G^+ is a generating function of (orbifold) Euler characters of Hurwitz spaces of holomorphic maps from Σ_g to Σ_G. This is explained in section 5 of [5] and reviewed in [4].

The complete expansion of the Z_G takes a similar form

$$Z_G = \sum_{m,n=0}^{\infty} \frac{N^{(m+n)(2-2G)}}{m!n!} \sum_{R\vdash m, S\vdash n} \sum_{s_1,t_1\cdots s_G,t_G\in S_m\times S_n} \delta_{m,n}\left(\Omega_{m,n}^{2-2G}\prod_{i=1}^{G} s_i t_i s_i^{-1} t_i^{-1}\right) \tag{1.9}$$

The $\delta_{m,n}$ is the delta function defined as in (1.6) but for $\mathbb{C}(S_m \times S_n)$. The "coupled Omega factor" $\Omega_{m,n}$ is much more intricate than the "chiral Omega factor" Ω_n but

$$\Omega_{m,n} = \Omega_m \Omega_n \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \tag{1.10}$$

The detailed formula and its interpretation in terms of wordsheet geometry is reviewed in section 4. The complete expansion is interpreted in terms of maps from worldsheets which have double points connecting two components which have branchings described by permutations in S_m and S_n respectively. The m sheets map holomorphically to the target and the n sheets map anti-holomorphically [2]. More precisely Z_G generates Euler characters of the appropriate moduli space of maps (see section 10 of [5]).

Recent work [6] has used, in the context of branes and anti-branes in the AdS dual of 4D super-Yang Mills, the Schur Weyl-duality between $\mathrm{U}(N)$ acting on $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ and the Brauer algebra $B(m, n)$. This algebra contains $\mathbb{C}(S_m \times S_n)$ as a sub-algebra, but also has additional generators corresponding to contractions between V and $\bar{V}$. Another important property of $B(m, n)$ is that there is a map $\Sigma : B(m, n) \to \mathbb{C}(S_{m+n})$. This map is not a homomorphism but maps the natural bilinear symmetric form on $B(m, n)$ to a bilinear symmetric form on the group algebra $\mathbb{C}(S_{m+n})$, which can be calculated in terms of Ω_{m+n}. The inversion of the form, which is useful in constructing projection operators in the Brauer algebra [7], is conveniently done using Ω_{m+n}^{-1}. This can be used to derive a formula for $DimR\bar{S}$ in terms of S_{m+n} data [6]. In this paper we will develop this further to derive a simple relation between $\Omega_{m,n}^{-1}$ and Ω_{m+n}^{-1}. We then describe the implications for the string interpretation of the $\frac{1}{N}$ expansion of 2DYM.

Section 2 derives the relation between $\Omega_{m,n}^{-1}$ and Ω_{m+n}^{-1}. In section 3 we use it to rewrite the complete $\frac{1}{N}$ expansion of 2DYM for $\Sigma(G = 2)$. In section 4 we give the geometrical interpretation. In section 5 we show that the same discussion carries over for $\Sigma(G)$. The complete expansion (1.9) can be interpreted in terms of holomorphic maps. As discussed in section 4, for the case $\Sigma(G = 1, B = 1)$ this is a straightforward consequence of the new formula for $\Omega_{m,n}^{-1}$. In general it requires a choice of cutting of $\Sigma(G)$ into components of Euler character -1 i.e 3-holed spheres or 1-holed tori.

JHEP06(2008)015

2. New dimension formula and Ω factors

In [6], we obtained a new formula for the coupled dimension,

$$\frac{1}{DimR\bar{S}} = \frac{1}{d_R^2 d_S^2}\left(\frac{m!n!}{(m+n)!}\right)^2 \sum_T \frac{d_T^2}{DimT} g(R,S;T) \tag{2.1}$$

We rewrite it using the formula for the Littlewood-Richardson (LR) coefficient

$$g(R,S;T) = \frac{1}{d_R d_S DimT} tr_{m+n}((p_R \circ p_S)p_T) \tag{2.2}$$

p_R is a projection operator in $\mathbb{C}(S_m)$ (see appendix A), p_S is in $\mathbb{C}(S_n)$ and p_T in $\mathbb{C}(S_{m+n})$. We also use

$$tr_{m+n}(\sigma) = N^{m+n}\delta_{m+n}(\Omega_{m+n}\sigma) \tag{2.3}$$

and

$$\frac{1}{(DimT)^2} = \left(\frac{(m+n)!}{N^{m+n}d_T}\right)^2 \frac{\chi_T(\Omega_{m+n}^{-2})}{d_T} \tag{2.4}$$

Then we have

$$\begin{aligned}
\frac{d_T^2}{DimT} g(R,S;T) &= \frac{d_T^2}{d_R d_S}\frac{1}{(DimT)^2} tr_{m+n}((p_R \circ p_S)p_T) \\
&= \frac{1}{d_R d_S}\left(\frac{(m+n)!}{N^{m+n}}\right)^2 \frac{\chi_T(\Omega_{m+n}^{-2})}{d_T} tr_{m+n}((p_R \circ p_S)p_T) \\
&= \frac{1}{d_R d_S}\left(\frac{(m+n)!}{N^{m+n}}\right)^2 tr_{m+n}(\Omega_{m+n}^{-2}(p_R \circ p_S)p_T) \\
&= \frac{1}{d_R d_S}\left(\frac{(m+n)!}{N^{m+n}}\right)^2 N^{m+n}\delta_{m+n}(\Omega_{m+n}^{-1}(p_R \circ p_S)p_T)
\end{aligned} \tag{2.5}$$

Therefore the formula (2.1) can be brought to the form

$$\begin{aligned}
\frac{1}{DimR\bar{S}} &= \sum_T \frac{1}{d_R^3 d_S^3}\frac{m!^2 n!^2}{N^{m+n}}\delta_{m+n}(\Omega_{m+n}^{-1}(p_R \circ p_S)p_T) \\
&= \frac{1}{d_R^3 d_S^3}\frac{m!^2 n!^2}{N^{m+n}}\delta_{m+n}(\Omega_{m+n}^{-1}(p_R \circ p_S))
\end{aligned} \tag{2.6}$$

where we have used $\sum_T p_T = 1$. This can be written as

$$\begin{aligned}
\frac{1}{DimR\bar{S}} &= \sum_{\sigma\in S_m}\sum_{\tau\in S_n} \frac{m!n!}{d_R^2 d_S^2 N^{m+n}}\chi_{R\otimes S}(\sigma^{-1}\otimes\tau^{-1})\delta_{m+n}((\sigma\otimes\tau)\Omega_{m+n}^{-1}) \\
&= \frac{m!n!}{d_R^2 d_S^2 N^{m+n}}\chi_{R\otimes S}(\Omega_{m+n}^{-1}|_{S_m\times S_n})
\end{aligned} \tag{2.7}$$

$\Omega_{m+n}^{-1}|_{S_m\times S_n}$ is calculated by expanding Ω_{m+n}^{-1} as an element of the group algebra of S_{m+n}, and then restricting to the subgroup $S_m \times S_n$. Comparing with the Gross-Taylor formula in terms of the coupled-Omega factor we find that

$$\Omega_{m,n}^{-1} = \Omega_{m+n}^{-1}|_{S_m\times S_n} \tag{2.8}$$

JHEP06(2008)015

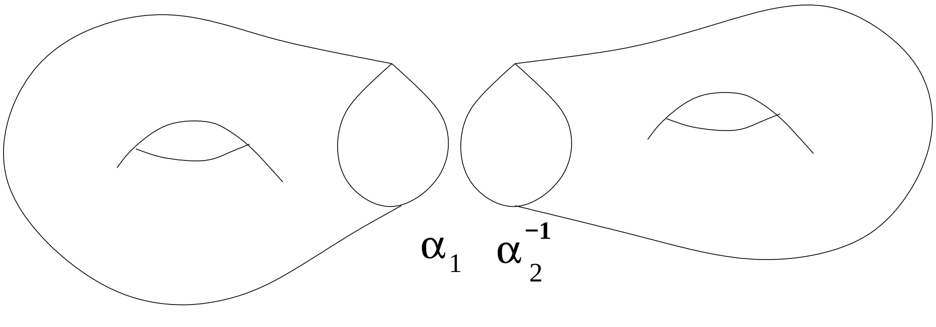

Figure 1: Genus two from gluing two copies of $\Sigma(G=1,B=1)$

As a simple example, we have

$$\begin{aligned}\Omega_2^{-1} &= \left(1-\frac{1}{N^2}\right)^{-1} - \frac{\sigma}{N}\left(1-\frac{1}{N^2}\right)^{-1}\\ \Omega_2^{-1}|_{S_1\times S_1} &= \left(1-\frac{1}{N^2}\right)^{-1}\\ &= \Omega_{1,1}^{-1}\end{aligned} \tag{2.9}$$

In appendix C, other examples involving $m+n=3,4$ are illustrated. An important point is that the relation (2.8) exists in the above simple form for $\Omega^{-1}_{m,n}$ and not $\Omega_{m,n}$. We cannot write $\Omega_{m,n}$ as a projection of Ω_{m+n}. We can get $\Omega_{m,n}$ from S_{m+n} data by using (2.8) and then inverting after the projection. This is related to the fact that the new geometrical interpretation of the complete $\frac{1}{N}$ expansion which we propose, works best for $\chi_{G,B}=2-2G-B\leq -1$.

3. Genus 2 target

3.1 Genus 2: partition function in terms of S_{m+n}

We will prove that the complete $1/N$ expansion of $Z_{G=2}$ is given by

$$\begin{aligned}Z_{G=2} = \sum_{m,n}\sum_{\alpha_1\in S_m\times S_n}&\frac{N^{-m-n}}{m!n!}\delta_{m+n}(\Omega^{-1}_{m+n}\Pi_1\alpha_1)\\ \times\sum_{\alpha_2\in S_m\times S_n}&\frac{N^{-m-n}}{m!n!}\delta_{m+n}(\Omega^{-1}_{m+n}\Pi_1\alpha_2^{-1})\\ \times\sum_{\gamma\in S_m\times S_n}&\delta_{m+n}(\alpha_1^{-1}\gamma\alpha_2\gamma^{-1})\end{aligned} \tag{3.1}$$

Here $\Pi_1=\sum_{s,t\in S_m\times S_n} sts^{-1}t^{-1}$. The above formula corresponds to gluing two 1-holed tori to get a genus 2-surface (see figure 1). Boundary partition functions with symmetric group data and their gluing is reviewed in appendix D.

We will also show

$$\begin{aligned} Z_{G=2} = & \sum_{m,n} \sum_{\alpha_1,\alpha_2,\alpha_3 \in S_m \times S_n} \frac{N^{-m-n}}{m!n!} \delta_{m+n}(\Omega_{m+n}^{-1} \alpha_1 \alpha_2 \alpha_3) \\ & \times \sum_{\beta_1,\beta_2,\beta_3 \in S_m \times S_n} \frac{N^{-m-n}}{m!n!} \delta_{m+n}(\Omega_{m+n}^{-1} \beta_1 \beta_2 \beta_3) \\ & \times \sum_{\gamma_1,\gamma_2,\gamma_3 \in S_m \times S_n} \prod_{i=1}^{3} \delta_{m+n}(\alpha_i^{-1} \gamma_i^{-1} \beta_i^{-1} \gamma_i) \end{aligned} \tag{3.2}$$

This corresponds to the gluing of two 3-holed spheres to get the genus 2 curve as in figure 2.

3.2 Derivations

Let us derive (3.1). We recall from [2] that

$$Z_{G=2} = \sum_{m,n} \frac{N^{-2(m+n)}}{m!n!} \delta_{m,n}(\Omega_{m,n}^{-2} \ \Pi_1^2) \tag{3.3}$$

We will use the abbreviation $S_{m,n} \equiv S_m \times S_n$. The delta function is over the group algebra of $S_{m,n}$. We can write this as

$$\begin{aligned} Z_{G=2} &= \sum_{m,n} \frac{N^{-2(m+n)}}{m!n!} \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \Omega_{m,n}^{-1} \ \Pi_1) \\ &= \sum_{m,n} \sum_{\alpha \in S_{m,n}} \frac{N^{-2(m+n)}}{m!n!} \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \alpha) \ \ \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \alpha^{-1}) \\ &= \sum_{m,n} \sum_{\alpha_1,\sigma_2 \in S_{m,n}} \frac{N^{-2(m+n)}}{m!n!} \ \ \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \alpha_1) \ \ \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \alpha_2^{-1}) \ \ \delta_{m,n}(\alpha_1^{-1} \alpha_2) \\ &= \sum_{m,n} \sum_{\alpha_1,\sigma_2,\gamma \in S_{m,n}} \frac{N^{-2(m+n)}}{(m!n!)^2} \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \gamma \alpha_1 \gamma^{-1}) \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \alpha_2^{-1}) \delta_{m,n}(\alpha_1^{-1} \alpha_2) \\ &= \sum_{m,n} \sum_{\alpha_1,\sigma_2,\gamma \in S_{m,n}} \frac{N^{-2(m+n)}}{(m!n!)^2} \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \alpha_1) \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \alpha_2^{-1}) \delta_{m,n}(\gamma \alpha_1^{-1} \gamma^{-1} \alpha_2) \\ &= \sum_{m,n} \sum_{\alpha_1 \in S_{m,n}} \frac{N^{-(m+n)}}{m!n!} \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \alpha_1) \\ & \qquad \times \sum_{\alpha_2 \in S_{m,n}} \frac{N^{-(m+n)}}{m!n!} \delta_{m,n}(\Omega_{m,n}^{-1} \ \Pi_1 \ \alpha_2^{-1}) \\ & \qquad \times \sum_{\gamma \in S_{m,n}} \ \ \delta_{m,n}(\gamma \alpha_1^{-1} \gamma^{-1} \alpha_2) \end{aligned} \tag{3.4}$$

The steps are each trivial. To get to the fourth equality we have used the fact that the $\Omega_{m,n}^{-1} \ \Pi_1$ is central in $S_{m,n}$. Now we will use a simple rewriting of $\delta_{m,n}(\Omega_{m,n}^{-1} B)$ where $B \in \mathbb{C}(S_m \times S_n)$. We know (2.8) that $\Omega_{m,n}^{-1}$ can be written as a projection of $\Omega_{m+n}^{-1} \in \mathbb{C}(S_{m+n})$. But when we have $\delta_{m,n}(\Omega_{m,n}^{-1} B)$ with $B \in \mathbb{C}(S_m \times S_n)$, then the delta function can be rewritten as $\delta_{m,n}(\Omega_{m,n}^{-1} B) = \delta_{m+n}(\Omega_{m+n}^{-1} B)$. The projection is being performed by

JHEP06(2008)015

the δ_{m+n} on the group algbera $\mathbb{C}(S_{m+n})$ and the fact that B belongs to the subalgebra. Using this observation we have

$$\begin{aligned} Z_{G=2} &= \sum_{m,n} \sum_{\alpha_1 \in S_{m,n}} \frac{N^{-(m+n)}}{m!n!} \delta_{m+n}(\Omega_{m+n}^{-1}\ \Pi_1\ \alpha_1) \\ &\quad \times \sum_{\alpha_2 \in S_{m,n}} \frac{N^{-(m+n)}}{m!n!} \delta_{m+n}(\Omega_{m+n}^{-1}\ \Pi_1\ \alpha_2^{-1}) \\ &\quad \times \sum_{\gamma \in S_{m,n}} \delta_{m+n}(\gamma\alpha_1^{-1}\gamma^{-1}\alpha_2) \end{aligned} \tag{3.5}$$

An important point is that the replacement $\delta_{m,n} \to \delta_{m+n}; \Omega_{m,n}^{-2} \to \Omega_{m+n}^{-2}$ cannot be done directly in (3.3) because Ω_{m+n}^{-2} involves multiplying $\Omega_{m+n}^{-1} \cdot \Omega_{m+n}^{-1}$ with both being viewed as elements in $\mathbb{C}(S_{m+n})$, whereas $\Omega_{m,n}^{-1} \cdot \Omega_{m,n}^{-1}$ is a multiplication in $\mathbb{C}(S_m \times S_n)$.

We will now demonstrate (3.2). Rewriting (3.3) by expanding Π_1

$$\begin{aligned} Z_{G=2} &= \sum_{s_1,t_1,s_2,t_2 \in S_{m,n}} \frac{N^{-2m-2n}}{m!n!} \delta_{m,n}(\Omega_{m,n}^{-2}\ s_1t_1s_1^{-1}t_1^{-1}\ s_2t_2s_2^{-1}t_2^{-1}) \\ &= \sum_{s_1,t_1,s_2,t_2 \in S_{m,n}} \frac{N^{-2m-2n}}{m!n!} \delta_{m,n}(\Omega_{m,n}^{-2}\ s_1s_2\ t_1s_1^{-1}t_1^{-1}\ t_2s_2^{-1}t_2^{-1}) \\ &= \sum_{s_1,t_1,s_2,t_2,s_3 \in S_{m,n}} \frac{N^{-2m-2n}}{m!n!} \delta_{m,n}(\Omega_{m,n}^{-1}\ s_1s_2s_3)\ \delta_{m,n}(\Omega_{m,n}^{-1}\ t_1s_1^{-1}t_1^{-1}\ t_2s_2^{-1}t_2^{-1}\ s_3^{-1}) \\ &= \sum_{s_i,t_i,u_i \in S_{m,n}} \frac{N^{-2m-2n}}{m!n!} \delta_{m,n}(\Omega_{m+n}^{-1}\ s_1s_2s_3)\ \delta_{m,n}(\Omega_{m+n}^{-1}\ u_1u_2u_3) \\ &\qquad \delta_{m,n}(t_1s_1^{-1}t_1^{-1}u_1^{-1})\ \delta_{m,n}(t_2s_2^{-1}t_2^{-1}u_2^{-1})\ \delta_{m,n}(u_3^{-1}s_3^{-1}) \\ &= \sum_{s_i,t_i,u_i \in S_{m,n}} \frac{N^{-m-n}}{m!n!} \delta_{m,n}(\Omega_{m+n}^{-1} s_1s_2s_3)\ \ \frac{N^{-m-n}}{m!n!} \delta_{m,n}(\Omega_{m+n}^{-1} u_1u_2u_3) \\ &\qquad \delta_{m,n}(t_1s_1^{-1}t_1^{-1}\ u_1^{-1})\delta_{m,n}(t_2s_2^{-1}t_2^{-1}\ u_2^{-1})\delta_{m,n}(t_3s_3t_3^{-1}\ u_3^{-1}) \end{aligned} \tag{3.6}$$

To get to the last equality, we have inserted $1 = \frac{1}{m!n!}\sum_{t_3 \in S_{m,n}} t_3t_3^{-1}$ into the inside of the last $\delta_{m,n}$ and redefined some variables. After a renaming $s_i \to \alpha_i, u_i \to \beta_i, t_i \to \gamma_i$, this proves (3.2).

3.3 Chiral form of complete $\frac{1}{N}$ expansion for $G = 2$

The chiral expansion of the partition function for genus 2 can be written in the same form as either (3.1) or (3.2). We use the label M for degree, and write for the chiral theory the form corresponding to the gluing of figure 2.

$$\begin{aligned} Z_{G=2}^{+} &= \sum_{M} \sum_{\alpha_i \in S_M} \frac{N^{-M}}{M!} \delta_M(\Omega_M^{-1}\alpha_1\alpha_2\alpha_3) \\ &\quad \times \sum_{\beta_i \in S_M} \frac{N^{-M}}{M!} \delta_M(\Omega_M^{-1}\beta_1^{-1}\beta_2^{-1}\beta_3^{-1}) \\ &\quad \times \sum_{\gamma_1,\gamma_2,\gamma_3 \in S_M} \prod_{i=1}^{3} \delta_M(\alpha_i^{-1}\gamma_i^{-1}\beta_i\gamma_i) \end{aligned} \tag{3.7}$$

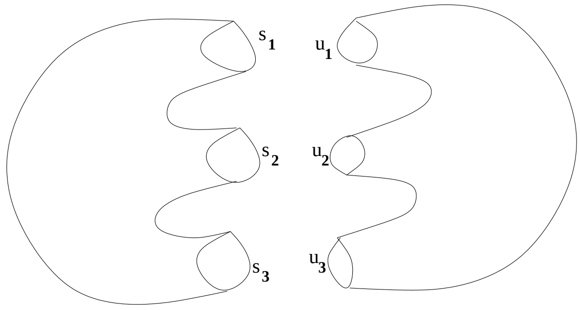

Figure 2: Genus two from gluing two copies of $\Sigma(G = 0, B = 3)$

To emphasize the similarity between (3.2) and (3.7) we can rewrite (3.2) as

$$\begin{aligned} Z_{G=2} = & \sum_M \sum_{\alpha_i \in S_M} \frac{N^{-M}}{M!} \delta_M(\Omega_M^{-1} \alpha_1 \alpha_2 \alpha_3) \\ & \times \sum_{\beta_i \in S_M} \frac{N^{-M}}{M!} \delta_M(\Omega_M^{-1} \beta_1^{-1} \beta_2^{-1} \beta_3^{-1}) \\ & \times \sum_{\gamma_1, \gamma_2, \gamma_3 \in S_M} \prod_{i=1}^{3} \delta_M(\alpha_i^{-1} \gamma_i^{-1} \beta_i \gamma_i) \\ & \times \sum_H \left(\frac{M!}{|H|} \right)^2 \prod_{i=1}^{3} \delta_M(\alpha_i \mathbf{1}_H) \delta_M(\beta_i \mathbf{1}_H) \delta_M(\gamma_i \mathbf{1}_H) \end{aligned} \tag{3.8}$$

where there is an additional sum over sub-groups $H = S_m \times S_n$ in $S_M \equiv S_{m+n}$, with $M \geq m, n \geq 0$. We have also defined $\mathbf{1}_H \equiv \sum_{\alpha \in H} \alpha$, the projector onto the symmetric irrep of H. The delta functions in the last line ensures that the permutations $\alpha_i, \beta_i, \gamma_i$ are in the subgroup H.

4. Two holomorphic descriptions of the non-chiral expansion

4.1 The Gross-Taylor coupled expansion: worldsheets, nodes and collision of branch points

The coupled Omega factor has an expansion [2]

$$\begin{aligned} \Omega_{m,n} &= \sum_{\sigma^+ \in S_m} \sum_{\sigma^- \in S_n} (\sigma^+ \otimes \sigma^-) P_{\sigma^+ \sigma^-} N^{C_{\sigma^+} + C_{\sigma^-} - (m+n)} \\ P_{\sigma^+ \sigma^-} &= \prod_{j=1} \sum_{m}^{min(k_j, l_j)} P_{k_j, l_j}(m) \frac{1}{N^{2m}} \\ P_{k_j, l_j}(m) &= \binom{k_j}{m} \binom{l_j}{m} m! (-j)^m \end{aligned} \tag{4.1}$$

In the above, j runs over the cycle lengths of σ^+, σ^-. k_j is the number of cycles of length j in σ^+ ; l_j is the number of cycles of length j in σ^-.

The coupled $\Omega_{m,n}$ factor contains a sum of permutations weighted by polynomials in $\frac{1}{N}$. In the chiral Ω_M factor, each permutation σ is weighted just by $N^{-b(\sigma)}$ where $b(\sigma)$ is the branching number of the permutation. In the case of the coupled $\Omega_{m,n}$ factor $\sigma \in S_m \times S_n$, written as $\sigma^+ \otimes \sigma^-$ to emphasize the product form, has a leading coefficient which is just the sum of branching numbers of the σ^+ and σ^-. The subleading terms have an elegant combinatoric interpretation discovered in [2]. They can be interpreted in terms of double points on the worldsheet, joining ramification points. The rule is that the double points can only connect ramification points of the same order, which can be zero. Each such double point is accompanied by a weight of $\frac{(-j)}{N^2}$ where j is the order of the ramification. The factor of $\frac{1}{N^2}$ accounts precisely for the change in Euler character of the worldsheet upon introduction of such a double point. A local model of the double point and the map was given in [5]. The branching described by $\sigma^+ \in S_m$ is taken to be that of a holomorphic map, and the branching described by σ^- is anti-holomorphic.

It is instructive to consider an orientation reversal on σ^- so that both σ^+, σ^- describe holomorphic maps. Then we can ask how the double points of the coupled expansion arise from the collision of branch points. For example consider a double cover over the sphere which is branched with two branch points each corresponding to the permutation (12). After collision the monodromy is just the identity permutation, with branching number zero. The Euler character of the worldsheet has not changed. This is consistent with a double point arising from the collision, which joins two points of trivial ramification. If we take a collision of branch points described by $(12\cdots j\ j+1)$ and $(j\ j+1\cdots 2j)$, the resulting permutation is $(1\cdots j)(j+1\cdots 2j)$ with branching number two less than the sum of branching numbers of the collising permutations, so a double point has been created. So all the double points of the type arising in the coupled expansion can occur from the collision of branch points, i.e at the boundaries of Hurwitz space. The most general collision of branch points can produce more complicated singularities, which do not all occur in the coupled expansion. For further comments on the collision of branch points and its relevance to the the identity (2.8) see the end of appendix C.

By generalizing the argument of the chiral sector to the coupled case (section 10 of [5]) it can be shown that the complete $\frac{1}{N}$ expansion of 2DYM computes Euler characters of holo-anti-holo maps, or after the orientation reversal just holomorphic maps, from worldsheets that can have double points according to the rules described above. This was called the moduli space of "degenerating coupled covers" in [5].

4.2 New holomorphic interpretation in the case $\Sigma(G = 1, B = 1)$

The simplest case where we can see the new interpretation based on (2.8) is for 2DYM on $\Sigma(G = 1, B = 1)$. In this case we have boundary observables specified by choosing an integer n and a conjugacy class T of $S_m \times S_n$ (see appendix D for a quick review and [4, 8] for more details)

$$Z(G = 1, B = 1; T) = \sum_{\alpha \in T} \sum_{s,t \in S_m \times S_n} \frac{N^{-m-n}}{m!n!} \delta_{m,n}(\Omega_{m,n}^{-1} sts^{-1}t^{-1}\alpha) \tag{4.2}$$

This partition function can be re-written as

$$Z(G=1,B=1;T)=\sum_{\alpha\in T}\sum_{s,t\in S_m\times S_n}\frac{N^{-m-n}}{m!n!}\delta_{m+n}(\Omega_{m+n}^{-1}sts^{-1}t^{-1}\alpha) \tag{4.3}$$

The first expression can be expanded and interpreted as an Euler character of the moduli space of "degenerating coupled covers" [5]. The second expression can be expanded

$$\begin{aligned}Z(G=1,B=1;T) &= \sum_{\alpha\in T}\sum_{s,t\in S_m\times S_n}\frac{N^{-m-n}}{m!n!}\sum_{L=0}^{\infty}d(-1,L)\\ &\times \sum'_{\sigma_1\cdots\sigma_L\in S_{m+n}}N^{-b(\sigma_1)-b(\sigma_2)-\cdots-b(\sigma_L)}\delta_{m+n}(\sigma_1\cdots\sigma_L sts^{-1}t^{-1}\alpha)\end{aligned} \tag{4.4}$$

Since the permutations $\sigma_1\cdots\sigma_L$ are in S_{m+n} the corresponding branch points can permute any of the $m+n$ sheets of the cover among each other. The power of N is consistent with these branching numbers. So these are holomorphic maps of degree $m+n$. The terms with fixed L can be interpreted in terms of a moduli space of holomorphic maps to $\Sigma(G=1,B=1)$ with L branch points with boundary permutation in the conjugacy class T. The cycle lengths of T correspond to winding numbers of strings at the boundary. The binomial factor $d(-1,L)=(-1)^L$ is the Euler character of the configuration space of L indistinguishable points on $\Sigma(G=1,B=1)$. Hence the $Z(G=1,B=1;T)$ is a generating function for the Euler character of the space of holomorphic maps with fixed string winding numbers at the boundary.

4.3 Holomorphic maps, sums over H-monodromies along markings: No singular worldsheets

We now describe the holomorphic interpretation for closed target spaces, for simplicity in the case of $\Sigma(G=2)$. It will be clear that the same ideas generalise to closed Riemann surfaces of any genus G.

The chiral expansion is a sum over M, which corresponds to the degree of the map from worldsheet Σ_g to the target $\Sigma(G)$. The chiral partition function $Z^+_{G=2}$ can be derived by gluing partition functions on one-holed tori or 3-holed spheres, e.g. (3.7). The final expression is independent of the choice of decomposition into $\chi=-1$ components as is manifest in (1.4). For each degree, the data at each boundary required to specify the boundary partition function is a conjugacy class in S_M. In the gluing procedure, we sum over all conjugacy classes in S_M and subsequently sum over M.

In the formulae developed above (3.1) (3.2), for the complete $\frac{1}{N}$ expansion, the gluing procedure is generalised. It involves summing over subgroups $H=S_m\times S_n$ of $S_{M=m+n}$. For each choice of H, we consider boundary partition functions depending on a conjugacy class in H, and the s,t monodromies in the subgroup as well. The boundary permutations and the s,t monodromies are summed over H. The branch points coming from expanding Ω^{-1}_{m+n} however are general permutations in S_M. The branch points can permute any of the

M sheets, so these are holomorphic maps of degree M. For example, from (3.1) we have

$$
\begin{aligned}
Z_{G=2} = \sum_{m,n} \sum_{\alpha_1 \in S_m \times S_n} \frac{N^{-m-n}}{m!n!} \sum_{L_1=0}^{\infty} \quad & \sum_{\sigma_1,\dots,\sigma_{L_1} \in S_{m+n}}^{\prime} d(-1,L_1) \;\; N^{-b(\sigma_1)-b(\sigma_2)\cdots-b(\sigma_{L_1})} \\
\times \sum_{\alpha_2 \in S_m \times S_n} \frac{N^{-m-n}}{m!n!} \sum_{L_2=0}^{\infty} \quad & \sum_{\tau_1,\tau_2,\dots\tau_{L_2} \in S_{m+n}}^{\prime} d(-1,L_2) \;\; N^{-b(\tau_1)-b(\tau_2)\cdots-b(\tau_{L_2})} \\
& \times \sum_{s,t \in S_m \times S_n} \delta_{m+n}(\sigma_1\sigma_2\cdots\sigma_{L_1} s t s^{-1} t^{-1} \alpha_1) \\
& \times \sum_{\tilde{s},\tilde{t} \in S_m \times S_n} \delta_{m+n}(\tau_1\tau_2\cdots\tau_{L_2} \tilde{s}\tilde{t}\tilde{s}^{-1}\tilde{t}^{-1}\alpha_2^{-1}) \\
& \quad \times \sum_{\gamma \in S_m \times S_n} \delta_{m+n}(\alpha_1^{-1}\gamma\alpha_2\gamma^{-1})
\end{aligned}
\tag{4.5}
$$

The binomial coefficients $d(-1,L_1) = (-1)^{L_1}$ and $d(-1,L_2) = (-1)^{L_2}$ in the expansion of the Ω_{m+n}^{-1} factors are Euler characters the configuration spaces of points on each $\chi = -1$ component (see [5] for a quick review of these Euler characters). Hence we can interpret in terms of the Euler character of a moduli space of holomorphic maps where the branch points can move over these components. There are no singular wordsheets in this interpretation. All the branch points coming from expanding the Ω factors are weighted with powers of N according to their branching number.

The complete partition function can be written in the suggestive form of an insertion, involving an additional sum over subgroups H of S_M, in the chiral partition function (3.8). In the next section we will find the higher genus analogs of the formulae in section 3. The possible implications of (3.8) in terms of observables in a topological string theory of holomorphic maps will be discussed in the general genus case in section 5.

To summarize, the complete $\frac{1}{N}$ expansion of 2DYM as given in (1.9) can be interpreted as a generating function of Euler characters of moduli spaces of holomorphic maps in two different ways. In one interpretation [2, 4], based on the formula for $\Omega_{m,n}^{-1}$ in [2], there are worldsheets with double points and branch points which can wander all over $\Sigma(G)$. The standard interpretation involves holomorphic and anti-holomorphic maps on different components joined at double points, but we can get a corresponding holomorphic moduli space by an orientation reversal on the anti-holomorphic component. A new interpretation using Ω_{m+n}^{-1} follows directly from unravelling the consequences of (2.8). In this interpretation there are no worldsheet double points and the branch points are free to move on the $\chi = -1$ components of a decomposition of $\Sigma(G)$ fixed by choosing some markings on the Riemann surface. By generalising the way we lift the gluing together of spacetime $\Sigma(G)$ along the markings to the gluing of spaces of maps from worldsheets to $\Sigma(G)$, we are able to get rid of the worldsheet double points. Since the expressions (3.1) (3.2) are derived from (1.9), it is clear that the generalized gluing is compatible with a well-defined partition function on $\Sigma(G)$, *independent of the choice of markings* which separate $\Sigma(G)$ into components of $\chi = -1$. The equivalence of different descriptions which give rise to the same expansion in the string coupling $g_s = \frac{1}{N}$ is reminiscent of T-duality. A natural question is whether

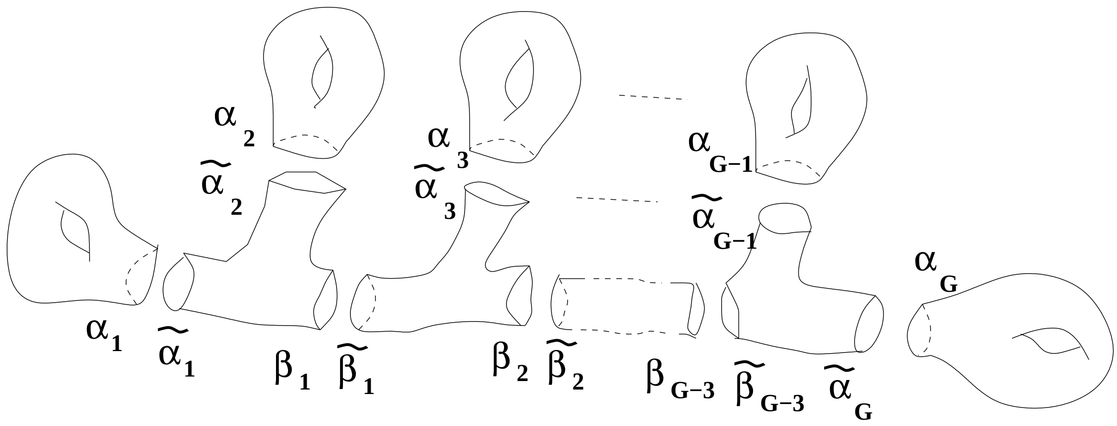

Figure 3: Genus G from gluing G copies of $\Sigma(G=1, B=1)$ and $G-2$ copies of $\Sigma(G=0, B=3)$

the equivalence of Euler characters of the two different moduli spaces described above is a duality that can be studied by physical methods from the worldsheet point of view.[1]

Given the crucial role played by the generalised gluing, it would be interesting to investigate the generalisation arising in (3.8), involving a summation over subgroups H of a group $\mathcal{G}$, in the context of general two dimensional topological field theories of a finite group $\mathcal{G}$. While $\mathcal{G} = S_M$ is of special interest in the large N expansion of 2DYM due to the Hurwitz connection between branched covers and S_M, other finite groups might be of interest in connection with the topological sector of CFTs with orbifold target spaces.

5. General genus

5.1 Z_G in terms of δ_{m+n}

The following fomula is derived in appendix B.1

$$\begin{aligned}
Z_G = \sum_{m,n} \sum_{\alpha_i \tilde{\alpha}_i, \beta_i, \tilde{\beta}_i \in S_m \times S_n} \prod_{i=1}^{G} \frac{N^{-m-n}}{m!n!} \delta_{m+n}(\Omega_{m+n}^{-1} \Pi_1 \alpha_i^{-1}) \\
\times \frac{N^{-m-n}}{m!n!} \delta_{m+n}(\Omega_{m+n}^{-1} \tilde{\alpha}_1 \tilde{\alpha}_2 \beta_1) \frac{N^{-m-n}}{m!n!} \delta_{m+n}(\Omega_{m+n}^{-1} \tilde{\beta}_1^{-1} \tilde{\alpha}_3 \beta_2) \\
\times \frac{N^{-m-n}}{m!n!} \delta_{m+n}(\Omega_{m+n}^{-1} \tilde{\beta}_2^{-1} \tilde{\alpha}_4 \beta_3) \cdots \frac{N^{-m-n}}{m!n!} \delta_{m+n}(\Omega_{m+n}^{-1} \tilde{\beta}_{G-3}^{-1} \tilde{\alpha}_{G-1} \tilde{\alpha}_G) \\
\times \sum_{\gamma_i, \epsilon_i \in S_m \times S_n} \prod_{i=1}^{G} \delta_{m+n}(\alpha_i \gamma_i \tilde{\alpha}_i^{-1} \gamma_i^{-1}) \prod_{i=1}^{G-3} \delta_{m+n}(\beta_i^{-1} \epsilon_i \tilde{\beta}_i \epsilon_i^{-1})
\end{aligned} \tag{5.1}$$

The corresponding choice of markings that separate $\Sigma(G)$ into $\chi = -1$ components is shown in figure 3. We have G copies of 1-holed tori glued to 3-holed spheres by permutations α_i. There are $G-2$ copies of the 3-holed spheres.

[1] We thank R. de Mello Koch for a discussion on this point.

JHEP06(2008)015

Another way to write the partition function employs the cutting of genus G into $2G-2$ copies of $\Sigma(G=0,B=3)$ as in figure 4.

$$
\begin{aligned}
Z_G =& \sum_{m,n} \sum_{s_i,t_i,u_i,v_i,w_i,\tilde{v}_i,\tilde{w}_i,\gamma_i,\epsilon_i \in S_m\times S_n} \frac{N^{-(m+n)}}{m!n!}\delta_{m+n}(\Omega_{m+n}^{-1}s_1s_2v_1)\frac{N^{-(m+n)}}{m!n!}\delta_{m+n}(\Omega_{m+n}^{-1}u_1u_2w_1)\\
&\frac{N^{-(m+n)}}{m!n!}\delta_{m+n}(\Omega_{m+n}^{-1}\tilde{v}_1^{-1}s_3v_2)\ \frac{N^{-(m+n)}}{m!n!}\delta_{m+n}(\Omega_{m+n}^{-1}\tilde{w}_1^{-1}u_3w_2)\\
&\qquad\vdots \qquad\qquad\qquad\qquad \vdots\\
&\frac{N^{-(m+n)}}{m!n!}\delta_{m+n}(\Omega_{m+n}^{-1}\tilde{v}_{G-3}^{-1}s_{G-1}v_{G-2})\frac{N^{-(m+n)}}{m!n!}\delta_{m+n}(\Omega^{-1}\tilde{w}_{G-3}^{-1}s_{G-1}w_{G-2})\\
&\frac{N^{-(m+n)}}{m!n!}\delta_{m+n}(\Omega_{m+n}^{-1}\tilde{v}_{G-2}^{-1}s_Gs_{G+1})\frac{N^{-(m+n)}}{m!n!}\delta_{m+n}(\Omega_{m+n}^{-1}\tilde{w}_{G-2}^{-1}u_Gu_{G+1})\\
&\prod_{i=1}^{G+1}\delta_{m+n}(u_it_is_i^{-1}t_i^{-1})\prod_{i=1}^{G-2}\delta_{m+n}(v_i^{-1}\gamma_i\tilde{v}_i\gamma_i^{-1})\prod_{i=1}^{G-2}\delta_{m+n}(w_i^{-1}\epsilon_i\tilde{w}_i\epsilon_i^{-1}) \qquad (5.2)
\end{aligned}
$$

We have introduced the gluing permutations γ_i, ϵ_i so that we get, for each component of Euler character -1 a boundary partition function with standard normalisation. We can therefore write

$$
\begin{aligned}
Z_G =& \sum_{m,n} \sum_{s_i,t_i,u_i,v_i,w_i,\tilde{v}_i,\tilde{w}_i,\gamma_i,\epsilon_i \in S_m\times S_n}\\
&Z(G=0,B=3;s_1s_2v_1)Z(G=0,B=3;u_1u_2w_1)\\
&Z(G=0,B=3;\tilde{v}_1^{-1}s_3v_2)Z(G=0,B=3;\tilde{w}_1^{-1}u_3w_2)\\
&\qquad\vdots \qquad\qquad\qquad\qquad \vdots\\
&Z(G=0,B=3;\tilde{v}_{G-3}^{-1}s_{G-1}v_{G-2})Z(G=0,B=3;\tilde{w}_{G-3}^{-1}s_{G-1}w_{G-2})\\
&Z(G=0,B=3;\tilde{v}_{G-2}^{-1}s_Gs_{G+1})Z(G=0,B=3;\tilde{w}_{G-2}^{-1}u_Gu_{G+1})\\
&\prod_{i=1}^{G+1}\delta_{m+n}(u_it_is_i^{-1}t_i^{-1})\prod_{i=1}^{G-2}\delta_{m+n}(v_i^{-1}\gamma_i\tilde{v}_i\gamma_i^{-1})\prod_{i=1}^{G-2}\delta_{m+n}(w_i^{-1}\epsilon_i\tilde{w}_i\epsilon_i^{-1}) \qquad (5.3)
\end{aligned}
$$

5.2 Chiral form of complete $\frac{1}{N}$ expansion for general G

The chiral partition function can also be written in a way to emphasize the construction of the genus G surface by gluing pants diagrams

$$
\begin{aligned}
Z_G^+ =& \sum_{M} \sum_{s_i,t_i,u_i,v_i,w_i,\tilde{v}_i,\tilde{w}_i,\gamma_i,\epsilon_i \in S_M} Z^+(G=0,B=3;s_1s_2v_1)Z^+(G=0,B=3;u_1u_2w_1)\\
&Z^+(G=0,B=3;\tilde{v}_1^{-1}s_3v_2)Z^+(G=0,B=3;\tilde{w}_1^{-1}u_3w_2)\\
&\qquad\vdots \qquad\qquad\qquad\qquad \vdots\\
&Z^+(G=0,B=3;\tilde{v}_{G-3}^{-1}s_{G-1}v_{G-2})Z^+(G=0,B=3;\tilde{w}_{G-3}^{-1}s_{G-1}w_{G-2})\\
&Z^+(G=0,B=3;\tilde{v}_{G-2}^{-1}s_Gs_{G+1})Z^+(G=0,B=3;\tilde{w}_{G-2}^{-1}u_Gu_{G+1})\\
&\prod_{i=1}^{G+1}\delta_M(u_it_is_i^{-1}t_i^{-1})\prod_{i=1}^{G-2}\delta_M(v_i^{-1}\gamma_i\tilde{v}_i\gamma_i^{-1})\prod_{i=1}^{G-2}\delta_M(w_i^{-1}\epsilon_i\tilde{w}_i\epsilon_i^{-1}) \qquad (5.4)
\end{aligned}
$$

JHEP06(2008)015

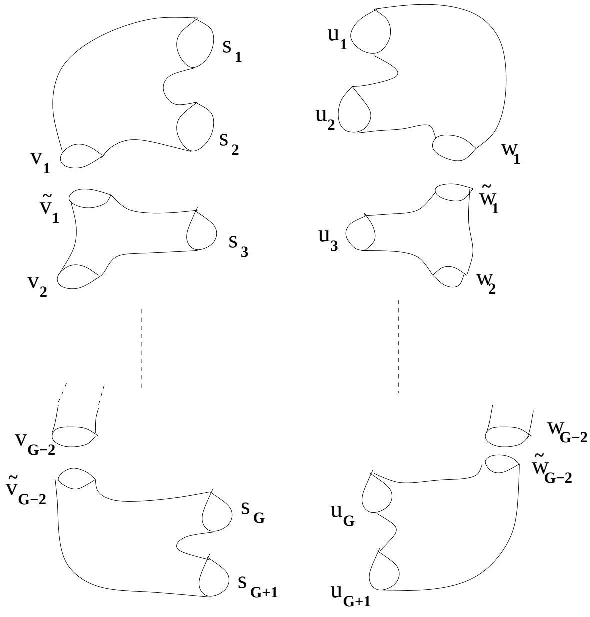

Figure 4: Genus G from gluing $2G-2$ copies of $\Sigma(G=0,B=3)$

To emphasize the similarity between the full partition function and the chiral one, we write

$$
\begin{aligned}
Z_G = \sum_M \sum_{s_i,t_i,u_i,v_i,w_i,\tilde{v}_i,\tilde{w}_i,\gamma_i,\epsilon_i \in S_M} & Z^+(G=0,B=3;s_1 s_2 v_1) Z^+(G=0,B=3;u_1 u_2 w_1) \\
& Z^+(G=0,B=3;\tilde{v}_1^{-1} s_3 v_2) Z^+(G=0,B=3;\tilde{w}_1^{-1} u_3 w_2) \\
& \qquad\qquad\qquad\qquad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
& Z^+(G=0,B=3;\tilde{v}_{G-3}^{-1} s_{G-1} v_{G-2}) Z^+(G=0,B=3;\tilde{w}_{G-3}^{-1} s_{G-1} w_{G-2}) \\
& Z^+(G=0,B=3;\tilde{v}_{G-2}^{-1} s_G s_{G+1}) Z^+(G=0,B=3;\tilde{w}_{G-2}^{-1} u_G u_{G+1}) \\
& \prod_{i=1}^{G+1} \delta_M(u_i t_i s_i^{-1} t_i^{-1}) \prod_{i=1}^{G-2} \delta_M(v_i^{-1}\gamma_i \tilde{v}_i \gamma_i^{-1}) \prod_{i=1}^{G-2} \delta_M(w_i^{-1}\epsilon_i \tilde{w}_i \epsilon_i^{-1}) \\
& \sum_H \left(\frac{M!}{|H|}\right)^{2G-2} \prod_{i=1}^{G+1} \delta_M(s_i \mathbf{1}_H)\delta_M(u_i \mathbf{1}_H)\delta_M(t_i \mathbf{1}_H) \\
& \prod_{i=1}^{G-2} \delta_M(v_i \mathbf{1}_H)\delta_M(\tilde{v}_i \mathbf{1}_H)\delta_M(w_i \mathbf{1}_H)\delta_M(\tilde{w}_i \mathbf{1}_H)\delta_M(\gamma_i \mathbf{1}_H)\delta_M(\epsilon_i \mathbf{1}_H)
\end{aligned}
\tag{5.5}
$$

The sum over H is the sum over $S_m \times S_n$ subgroups, with $0 \leq m,n \leq M$ and $m+n=M$. We have defined $\mathbf{1}_H \equiv \sum_{\sigma\in H}\sigma$. When H is restricted to be S_M, i.e $(m,n)=(0,M)$ or $(M,0)$, we have the standard chiral partitions functions. The expression suggests an interpretation within a topological string theory of holomorphic maps with target $\Sigma(G)$, of the complete $\frac{1}{N}$ expansion of the 2DYM partition function, in terms of the insertion of an appropriate observable corresponding to

$$\sum_H \prod_{i=1}^{G+1} \delta_M(s_i \mathbf{1}_H)\delta_M(u_i \mathbf{1}_H)\delta_M(t_i \mathbf{1}_H)$$
$$\times \prod_{i=1}^{G-2} \delta_M(v_i \mathbf{1}_H)\delta_M(\tilde{v}_i \mathbf{1}_H)\delta_M(w_i \mathbf{1}_H)\delta_M(\tilde{w}_i \mathbf{1}_H)\delta_M(\gamma_i \mathbf{1}_H)\delta_M(\epsilon_i \mathbf{1}_H) \quad (5.6)$$

Identifying such an obervable in terms of classes on the Hurwitz space of holomorphic maps, or in terms of the pull-back of these classes to the moduli space of worldsheet complex structures would be the next step in developing the holomorphic description of the full $\frac{1}{N}$ expansion of 2DYM. If the classes on $\overline{\mathcal{M}}_{g,n}$ can be expressed in terms of the Mumford-Morita classes with intersection numbers computed by 2D quantum gravity [25], this could lead to new connections between 2DYM and integrable equations. This would give a concrete way to explicitly compute the terms in the $\frac{1}{N}$ expansion directly from wordsheet topological string methods.

6. Summary and outlook

By developing results in [6] we have found an expression (2.8) for the coupled inverse Omega factor $\Omega^{-1}_{m,n}$ of [2] in terms of a projection of the chiral inverse Omega factor Ω^{-1}_{m+n}. The latter has a simple interpretation in terms of branch points. This has allowed us to write the complete $\frac{1}{N}$ expansion of the partition function of 2DYM theory, with $\mathrm{SU}(N)$ gauge group, as an insertion of an observable in the chiral partition function (see (3.8) and (5.5)). The chiral form of the complete $\frac{1}{N}$ expansion uses a choice of markings which separate the target space $\Sigma(G)$ into components of Euler character -1. i.e 3-holed spheres or 1-holed tori. The partition function does not depend on the choice of markings. The difference between the chiral expansion and the complete one is simply in the choice of gluing factors at the markings. The complete expansion has an additional sum over subgroups $S_m \times S_n$ of S_M where $M = m+n$ is the degree of the map from worldsheet to target. The geometrical interpretation of the coupled $\Omega^{-1}_{m,n}$ factor involves worldsheets with double points which can arise from collision of branch points. The expression in terms of the chiral Ω^{-1}_{m+n} factor allows a geometrical interpretation with smooth worldsheets $\Sigma(g)$ without double points mapping to the target space. In particular we have an equality of the Euler character of a space of holomorphic maps from worldsheets which can have nodes and the Euler character of a space of holomorphic maps from smooth worldsheets.

Several extensions of these results are worth investigating. Incorporating finite area A or changing the gauge group from $\mathrm{SU}(N)$ to $\mathrm{U}(N)$ can be done trivially. The latter involves an extra sum over a $\mathrm{U}(1)$ charge. The dimensions of irreps are unaffected by tensoring with the $\mathrm{U}(1)$ representations, so there is no non-trivial modification. In the bulk of this paper we have used $\mathrm{SU}(N)$ rather than $\mathrm{U}(N)$ because the main points about the chiral reformulation can be made at zero area and zero theta parameter in the former case. In the case of $\mathrm{U}(N)$ we have to include the area or the theta parameter to control the infinite sum over $\mathrm{U}(1)$ irreps. The generalisation of the large N expansion of 2DYM for the gauge groups $O(N)$ and $\mathrm{Sp}(N)$ is known [9, 10]. This expansion involves worldsheets with nodes, and the additional feature of possible non-orientability. Is there a rewriting of the Ω factors of $O(N), \mathrm{Sp}(N)$ which allows us to map the partition function to one involving worldsheets that do not involve nodes ? In the $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ theory, there are non-perturbative sectors which have an interpretation of terms of splitting fermi seas [11]. These theories also admit q-deformations which have a string interpretation in terms of strings with Calabi-Yau targets which are direct sums of line bundles over $\Sigma(G)$ [12] (see also subsequent work [13 − 15]). The chiral $\frac{1}{N}$ (more precisely $1/[N]$ where $[N]$ is a q-number) expansion for these theories have been worked out in terms of Hecke algebras [16]. It would be interesting to work out the chiral formulation of the full expansion in the q-deformed and non-perturbative sectors. An obvious question is to find how to express the observable inserted in (3.8) (5.5) in terms of the balanced topological string proposed as the worldsheet string for chiral 2DYM ([5, 17]). The chiral-anti-chiral split of 2D Yang Mills has also been interpreted in terms of the OSV conjecture for black hole entropy. It would be interesting to explore possible implications of our holomorphic reformulation for 2dYM in that context [18].

The result (2.8) has been found using Brauer algebras which have been useful in diagonalisation problems of the CFT-metric on gauge invariant Matrix operators in four dimensional $N = 4$ SYM gauge theory. These diagonalisation problems have been useful [19] in mapping gauge theory states to AdS-spacetime states such as 3-brane configurations (giant gravitons). Brauer algebras arise in the case where we have both branes and anti-branes [6]. The map Σ which has been crucial in developing formulae for projectors in the Brauer algebra [6], is also used to map projectors in $\mathbb{C}(S_{m+n})$ to Brauer elements in [20]. The appearance of the same algebraic structures in describing strings in the string theory dual of 2DYM and 3-branes in the string theory dual of 4DSYM suggests that the geometrical lessons of 2DYM will also have consequences for 4DSYM.

We expect that the reformulation of the complete (coupled) expansion in terms of the chiral theory can lead to a deeper mathematical understanding of the large N expansion of 2DYM theory. We venture some speculative ideas along these lines. From a physical perspective we want to understand, in generality, the relation between 2DYM for $\Sigma(G)$ and Matrix models such as the Kontsevich Matrix model [22] which can exhibit the relation with the geometry of the compactified moduli space of worldsheet complex structures $\overline{\mathcal{M}}_{g,n}$. There are some early attempts in this direction [21]. Certain discrete counting problems related to Hurwitz spaces for sphere target, some of which were considered in the context of 2D Yang Mills [23], have been mapped to integrals over the compactified moduli spaces of complex structures $\overline{\mathcal{M}_{g,n}}$ [24] and results from 2D gravity [25] have allowed explicit computations. In fact it is known the chiral 2DYM computes an Euler character of Hurwitz spaces [5]. It should be possible to express this Euler character in terms of integrals of cohomology classes over $\overline{\mathcal{M}}_{g,n}$. This should lead to a better understanding of how to express the observables inserted in (5.5), which give the complete $\frac{1}{N}$ expansion of 2DYM, in terms of classes on Hurwitz spaces and in turn on $\overline{\mathcal{M}}_{g,n}$. In line with other recent mathematical developments related to A-model topological strings [26, 27], the construction of the appropriate classes on Hurwitz space and $\overline{\mathcal{M}}_{g,n}$ should probably proceed by

JHEP06(2008)015

first introducing the more complicated compactification of stable maps for Gromov-Witten theory with $\Sigma(G)$ target space, and then introducing virtual classes whose integrals are simpler. In such a scenario, the two moduli spaces of equal Euler character described in section 4, have to be understood as the localization loci of virtual classes in the stable compactification. Research on these avenues would lead to new connections between integrable hierachies and the large N expansion of 2DYM. For the case of sphere target, the topological σ-models are already known to be related to a matrix model and integrable hierarchies [28], with conjectural relations for more general target [29]. An extension of analogous results to define a Matrix Model of Euler characters of holomorphic maps related to the balanced topological strings [5, 17] (and supplemented with the worldheet versions of the observable (5.5)) which generate the $\frac{1}{N}$ expansion of 2DYM would set the stage for a quantitative understanding of these strings for more general target spaces.

Acknowledgments

We thank Robert de Mello Koch, Gregory Moore, Sunil Mukhi, Rodolfo Rosso for useful discussions/correspondence. SR is supported by a STFC Advanced Fellowship and in part by the EC Marie Curie Research Training Network MRTN-CT-2004-512194. YK is supported by STFC grant PP/D507323/1 "String theory, gauge theory and gravity".

A. Useful formulae

The projector used in section 2 is

$$p_R = \frac{d_R}{m!} \sum_{\sigma} \chi_R(\sigma)\sigma \tag{A.1}$$

The appearance of Π_1 in large N expansions of 2DYM stems from

$$\left(\frac{m!}{d_R}\right)^2 = \sum_{s,t \in S_m} \frac{\chi_R(sts^{-1}t^{-1})}{d_R} \tag{A.2}$$

The delta function used extensively in 2DYM has a character expansion

$$\frac{1}{n!} \sum_{R} d_R \chi_R(\rho) = \delta(\rho) \tag{A.3}$$

The relation between dimensions and the inverse Ω factor is

$$\frac{1}{dimR} = \frac{m!}{N^m} \frac{\chi_R(\Omega_m^{-1})}{d_R^2} \tag{A.4}$$

Useful formulae for manifolds with boundary are

$$\begin{aligned} tr(\sigma U) &= \sum_R \chi_R(\sigma)\chi_R(U) \\ \int dU \chi_R(U)\chi_S(U^\dagger) &= \delta_{\rm RS} \end{aligned} \tag{A.5}$$

B. Derivations for genus G

For any decomposition of genus G into pants and one-holed tori we can write the full partition function in a way that reflects the choice of decomposition. Then going from chiral to full theory involves inserting the sum over subgroups and the projections of all the permutations using the symmetric projector of the subgroup.

We can write the genus G answer from [2]

$$
\begin{aligned}
Z_G &= \sum_{m,n} \frac{N^{(m+n)(2-2G)}}{m!n!} \delta_{m,n}(\Omega_{m,n}^{2-2G} \Pi_1^G) \\
&= \sum_{m,n} \frac{N^{(m+n)(2-2G)}}{m!n!} \sum_{\alpha_1 \cdots \alpha_G \in S_m \times S_n} \delta_{m,n}((\Omega_{m,n}^{-1})^{G-2} \prod_i \alpha_i) \prod_{i=1}^{G} \delta_{m,n}(\Omega_{m,n}^{-1} \Pi_1 \alpha_i^{-1}) \\
&= \sum_{m,n} \frac{N^{(m+n)(2-2G)}}{m!n!} \sum_{\alpha_1 \cdots \alpha_G \in S_m \times S_n} \prod_{i=1}^{G} \delta_{m,n}(\Omega_{m,n}^{-1} \Pi_1 \alpha_i^{-1}) \\
&\quad \sum_{\beta_1 \cdots \beta_{G-3} \in S_m \times S_n} \delta_{m,n}(\Omega_{m,n}^{-1} \alpha_1 \alpha_2 \beta_1) \delta_{m,n}(\Omega_{m,n}^{-1} \beta_1^{-1} \alpha_3 \beta_2) \\
&\qquad \delta_{m,n}(\Omega_{m,n}^{-1} \beta_2^{-1} \alpha_4 \beta_3) \cdots \delta_{m,n}(\Omega_{m,n}^{-1} \beta_{G-3} \alpha_{G-1} \alpha_G) \\
&= \sum_{m,n} \frac{N^{(m+n)(2-2G)}}{m!n!} \sum_{\alpha_1 \cdots \alpha_G \in S_m \times S_n} \prod_{i=1}^{G} \delta_{m+n}(\Omega_{m+n}^{-1} \Pi_1 \alpha_i^{-1}) \\
&\quad \sum_{\beta_1 \cdots \beta_{G-3} \in S_m \times S_n} \delta_{m+n}(\Omega_{m+n}^{-1} \alpha_1 \alpha_2 \beta_1) \delta_{m+n}(\Omega_{m+n}^{-1} \beta_1^{-1} \alpha_3 \beta_2) \\
&\qquad \delta_{m+n}(\Omega_{m+n}^{-1} \beta_2^{-1} \alpha_4 \beta_3) \cdots \delta_{m+n}(\Omega_{m+n}^{-1} \beta_{G-3} \alpha_{G-1} \alpha_G)
\end{aligned}
\tag{B.1}
$$

After manipulating so that each delta functions contains a simgle power of Ω^{-1} we can write in terms of the Ω_{m+n}^{-1} leaving the δ to do the projection.

We can also re-write in terms of any decomposition of the genus G into 3-holed spheres, by imitating steps analogous to (3.6)

$$
\begin{aligned}
Z_G &= \sum_{m,n} \frac{N^{(2-2G)(m+n)}}{m!n!} \sum_{s_1,t_1 \cdots s_G,t_G \in S_m \times S_n} \delta_{m,n}\left(\Omega_{m,n}^{2-2G}\ s_1 t_1 s_1^{-1} t_1^{-1}\ s_2 t_2 s_2^{-1} t_2^{-1} \cdots s_G t_G s_G^{-1} t_G^{-1}\right) \\
&= \sum_{m,n} \frac{N^{(2-2G)(m+n)}}{m!n!} \sum_{s_1,t_1 \cdots s_{G+1},t_{G+1} \in S_m \times S_n} \delta_{m,n}(\Omega_{m,n}^{1-G}\ s_1 s_2 \cdots s_G\ s_{G+1}) \\
&\quad \delta_{m,n}(\Omega_{m,n}^{1-G}\ t_1 s_1^{-1} t_1^{-1}\ t_2 s_2^{-1} t_2^{-1} \cdots t_G s_G^{-1} t_G^{-1} s_{G+1}^{-1}) \\
&= \sum_{m,n} \frac{N^{(2-2G)(m+n)}}{(m!n!)^2} \sum_{s_i,t_i,u_i \in S_m \times S_n} \delta_{m,n}(\Omega_{m,n}^{1-G}\ s_1 s_2 \cdots s_G s_{G+1}) \delta_{m,n}(\Omega_{m,n}^{1-G} u_1 u_2 \cdots u_G u_{G+1}) \\
&\quad \prod_{i=1}^{G+1} \delta_{m,n}(u_i t_i s_i^{-1} t_i^{-1}) \\
&= \sum_{m,n} \frac{N^{(2-2G)(m+n)}}{(m!n!)^2} \sum_{s_i,t_i,u_i} \sum_{v_1,w_1 \cdots v_{G-2},w_{G-2} \in S_m \times S_n} \delta_{m,n}(\Omega_{m,n}^{-1} s_1 s_2 v_1) \quad \delta_{m,n}(\Omega_{m,n}^{-1} u_1 u_2 w_1) \\
&\quad \delta_{m,n}(\Omega_{m,n}^{-1} v_1^{-1} s_3 v_2) \quad \delta_{m,n}(\Omega_{m,n}^{-1} w_1^{-1} u_3 w_2) \\
&\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

$$
\begin{aligned}
&\delta_{m,n}(\Omega_{m,n}^{-1}v_{G-3}^{-1}s_{G-1}v_{G-2})\delta_{m,n}(\Omega_{m,n}^{-1}w_{G-3}^{-1}s_{G-1}w_{G-2})\\
&\delta_{m,n}(\Omega_{m,n}^{-1}v_{G-2}^{-1}s_G s_{G+1})\delta_{m,n}(\Omega_{m,n}^{-1}w_{G-2}^{-1}u_G u_{G+1})\\
&\prod_{i=1}^{G+1}\delta_{m,n}(u_i t_i s_i^{-1}t_i^{-1})\\
=\sum_{m,n}\sum_{s_i,t_i,u_i,v_i,w_i,\tilde{v}_i,\tilde{w}_i,\gamma_i,\epsilon_i\in S_m\times S_n}&\frac{N^{-(m+n)}}{(m!n!)}\delta_{m,n}(\Omega_{m,n}^{-1}s_1s_2v_1)\frac{N^{-(m+n)}}{(m!n!)}\delta_{m,n}(\Omega_{m,n}^{-1}u_1u_2w_1)\\
&\frac{N^{-(m+n)}}{(m!n!)}\delta_{m,n}(\Omega_{m,n}^{-1}\tilde{v}_1^{-1}s_3v_2)\quad\frac{N^{-(m+n)}}{(m!n!)}\delta_{m,n}(\Omega_{m,n}^{-1}\tilde{w}_1^{-1}u_3w_2)\\
&\qquad\qquad\qquad\qquad\vdots\qquad\qquad\qquad\qquad\qquad\vdots\\
&\frac{N^{-(m+n)}}{(m!n!)}\delta_{m,n}(\Omega_{m,n}^{-1}\tilde{v}_{G-3}^{-1}s_{G-1}v_{G-2})\frac{N^{-(m+n)}}{(m!n!)}\delta_{m,n}(\Omega^{-1}\tilde{w}_{G-3}^{-1}s_{G-1}w_{G-2})\\
&\frac{N^{-(m+n)}}{(m!n!)}\delta_{m,n}(\Omega_{m,n}^{-1}\tilde{v}_{G-2}^{-1}s_G s_{G+1})\frac{N^{-(m+n)}}{(m!n!)}\delta_{m,n}(\Omega_{m,n}^{-1}\tilde{w}_{G-2}^{-1}u_G u_{G+1})\\
&\prod_{i=1}^{G+1}\delta_{m,n}(u_i t_i s_i^{-1}t_i^{-1})\prod_{i=1}^{G-2}\delta_{m,n}(v_i^{-1}\gamma_i\tilde{v}_i\gamma_i^{-1})\prod_{i=1}^{G-2}\delta_{m,n}(w_i^{-1}\epsilon_i\tilde{w}_i\epsilon_i^{-1})
\end{aligned}
\tag{B.2}
$$

Now that all the Ω^{-1} factors are sitting in separate delta functions in $S_m \times S_n$ along with permutations within that subgroup, we may obtain (5.2).

C. Omega factors

In this section, we show explicit forms of Ω_{m+n}^{-1} and $\Omega_{m,n}^{-1}$ for some examples. One way to calculate Ω_{m+n}^{-1} is to solve $\Omega_{m+n}\Omega_{m+n}^{-1} = 1$. Another useful way is to use

$$
\Omega_{m+n}^{-1} = \frac{N^{m+n}}{((m+n)!)^2}\sum_{T\vdash(m+n)}\frac{d_T^2}{DimT}\chi_T(\sigma)\sigma \tag{C.1}
$$

which was used in [6] to obtain the dual of Brauer algebra elements with respect to a bilinear form.

When $m + n = 3$, the omega factor is given by

$$
\Omega_3 = 1 + \frac{1}{N}T_{[2,1]} + \frac{1}{N^2}T_{[3]} \tag{C.2}
$$

The indices written as subscripts of T denote the cycle lengths of the conjugacy class. The inverse of this is calculated using the above formula as

$$
\Omega_3^{-1} = \frac{N^2}{(N^2-1)(N^2-4)}\left(N^2 - 2 - NT_{[2,1]} + 2T_{[3]}\right) \tag{C.3}
$$

By projecting this to the subgroup $S_2 \times S_1$, we get

$$
\Omega_3^{-1}|_{S_2\times S_1} = \Omega_{2,1}^{-1} = \frac{N^2}{(N^2-1)(N^2-4)}\left(N^2 - 2 - Ns_1\right) \tag{C.4}
$$

We can easily check $\Omega_{2,1}^{-1}\Omega_{2,1} = 1$ using

$$
\Omega_{2,1} = 1 - \frac{2}{N^2} + \frac{1}{N}s_1 \tag{C.5}
$$

JHEP06(2008)015

Another example is the case of $m+n=4$, where the inverse of the omega factor is

$$\Omega_4^{-1} = \frac{N^2}{(N^2-1)(N^2-4)(N^2-9)} \times \big(N^4 - 8N^2 + 6 - N(N^2-4)T_{[2,1^2]} + (2N^2-3)T_{[3,1]} -5NT_{[4]} + (N^2+6)T_{[2,2]}\big) \tag{C.6}$$

In this case, we can consider two projections to $S_3 \times S_1$ and $S_2 \times S_2$, which give

$$\Omega_4^{-1}|_{S_3\times S_1} = \Omega_{3,1}^{-1} = \frac{N^2}{(N^2-1)(N^2-4)(N^2-9)} \times \times \big(N^4 - 8N^2 + 6 - N(N^2-4)T_{[2,1]} + (2N^2-3)T_{[3]}\big) \tag{C.7}$$

and

$$\Omega_4^{-1}|_{S_2\times S_2} = \Omega_{2,2}^{-1} = \frac{N^2}{(N^2-1)(N^2-4)(N^2-9)} \times \times \big(N^4 - 8N^2 + 6 - N(N^2-4)(s+\bar{s}) + (N^2+6)s\bar{s}\big) \tag{C.8}$$

These can also be checked using

$$\Omega_{3,1} = 1 - \frac{3}{N^2} + \frac{1}{N}\left(1 - \frac{1}{N^2}\right)T_{[2,1]} + \frac{1}{N^2}T_{[3]} \tag{C.9}$$

and

$$\Omega_{2,2} = 1 - \frac{4}{N^2} + \frac{2}{N^4} + \frac{1}{N}(s+\bar{s}) + \frac{1}{N^2}\left(1 - \frac{2}{N^2}\right)s\bar{s} \tag{C.10}$$

Here s is the transposition in the left S_2 factor, while $\bar{s}$ is the transposition in the right S_2 factor.

According to (2.8) and the above explicit formulae, all the restrictions on the nature of the double points in $\Omega_{m,n}$ reviewed in section 4.1 are encoded in the inversion of Ω_{m+n} to give Ω_{m+n}^{-1}. This uses the properties of group multiplication inside S_{m+n} followed by projection to $S_m \times S_n$. It is not a straightforward relation such as saying that the double points of the coupled expansion follow from all those arising in the collision of branch points as encoded in symmetric group multiplication. For example general products of permutations can give nodes connecting different types of cycles. Consider multiplications in S_4 such as $(132)(1234) = (1)(2)(34)$. The counting of branching numbers implies that after such a collision there are two nodes degenerated at one point. Such nodes do not arise the coupled expansion. The above multiplication does however enter the relation between Ω_4^{-1} and $\Omega_{2,2}^{-1}$. If the reduction is to $S_{\{1,2\}} \times S_{\{3,4\}}$ then it seems we have tubes connecting 2-cycles to 1-cycles. This not so. We can think of $(1)(2)(34)$ as a product of trivial permutation with the $(1)(2)(34)$. We associate $\frac{1}{N^2}$ with the trivial permutation. In other words in expanding $\Omega_{m,n}^{-1} = (1+\tilde{\Omega}_{m,n})^{-1}$ the term corresponding to $N^{-5}(132)(1234)$ in the expansion of Ω_4^{-1} comes from the $(\tilde{\Omega}_{m,n})^2$ and not from $\tilde{\Omega}_{m,n}$.

D. Gluing manifolds with boundary

Gauge theory partition functions are defined as a function of the boundary holonomy U which lives in the gauge group $\mathrm{U}(N)$. They can be calculated exactly. Consider, for example, the case $\Sigma(G=1,B=1)$.

$$Z(G=1,B=1;U) = \sum_R \frac{1}{dimR} \chi_R(U) \tag{D.1}$$

To get observables appropriate for a string interpretation in the chiral expansion we choose a positive integer M and a conjugacy class T in S_M.

$$\begin{aligned} Z^+(G=1,B=1;T) &\equiv \int dU Z^+(U) \, \frac{1}{M!} \sum_{\alpha\in T} \sum_R \chi_R(\alpha)\chi_R(U^\dagger) \\ &= \frac{N^{-M}}{M!} \sum_{s,t\in S_M} \sum_{\alpha\in T} \delta_M(\Omega_M^{-1} sts^{-1}t^{-1}\alpha) \end{aligned} \tag{D.2}$$

The delta function is defined over the group algebra of S_M. This can be interpreted as a sum over covering spaces of $\Sigma(G=1,B=1)$ subject to the constraints that the permutation of the sheets upon going on a path round the boundary is in the conjugacy class T. After expanding the Ω factors as in [5]

$$\begin{aligned} Z^+(G=1,B=1;T) = \frac{N^{-M}}{M!} \sum_{L=0}^{\infty} \sum'_{\sigma_1,\ldots,\sigma_L\in S_M} \sum_{s,t\in S_M} \sum_{\alpha\in T} (-1)^L \\ \times \left(\frac{1}{N}\right)^{\sum_{j=1}^L (M-C_{\sigma_j})} \delta_M(\sigma_1\cdots\sigma_L sts^{-1}t^{-1}\alpha) \end{aligned} \tag{D.3}$$

The factor $(-1)^L$ is the Euler character of the L-dimensional configuration space of L branch points on $\Sigma(G=1,B=1)$, hence the interpretation as an Euler character of the moduli space of branched covers. This is explained in detail in [5, 4]

In the non-chiral theory, we choose two integers m,n and a conjugacy class T in $S_m\times S_n$. and we multiply $Z(U)$ with characters $\sum_{\sigma\in T}\chi_{R\otimes S}(\sigma)\chi_{R\bar{S}}(U^\dagger)$.

$$\begin{aligned} Z(G=1,B=1;T) &= \sum_{\alpha\in T} Z(G=1,B=1;\alpha) \\ &= \sum_{\alpha\in T} \frac{N^{-m-n}}{m!n!} \sum_{s,t\in S_m\times S_n} \delta_{m,n}(\Omega_{m,n}^{-1} sts^{-1}t^{-1}\alpha) \end{aligned} \tag{D.4}$$

The delta function is defined over the group algebra of $S_m\times S_n$. Given the formula for $\Omega_{m,n}$ in (4.1) this can be interpreted in terms branched coverings from wordsheets made of pairs of surfaces joined at double points. One component of the pair maps holomorphically, the other maps anti-holomprphically. After expanding the $\Omega_{m,n}$ factor we can interpret $Z(G=1,B=1;T)$ as an Euler character of a moduli space of "coupled maps." By using an orientation reversal, the coupled maps are nothing but degenerated holomorphic maps.

The gluing of partition functions is done by integrating over the $\mathrm{U}(N)$ holonomy U along a common boundary. This can be translated into a rule for how to glue the partition functions with boundary data in terms of symmetric groups. In the chiral theory, the rule is to sum over all possible T for all M. We describe the gluing of two copies of $\Sigma(G=1,B=1)$ to get $\Sigma(G=2,B=0)$

$$Z^{+}(G=2)=\sum_{M}\sum_{\alpha_1,\beta_1\in S_M}Z^{+}(G=1,\alpha_1)Z^{+}(G=1,\beta_1^{-1})\\ \times\sum_{\gamma\in S_M}\delta_M(\alpha_1^{-1}\gamma\beta_1\gamma^{-1}) \tag{D.5}$$

We expect the gluing of boundary partition functions for two copies of $Z_{G=0,B=3}$ to give $Z(G=2)$ and indeed the following equality holds

$$Z^{+}(G=2)=\sum_{M}\sum_{\alpha_i,\beta_i\in S_M}Z^{+}(G=0,B=3;\alpha_1,\alpha_2,\alpha_3)Z^{+}(G=0,B=3;\beta_1^{-1},\beta_2^{-1},\beta_3^{-1})\\ \times\prod_{i=1}^{3}\sum_{\gamma_i}\delta_M(\alpha_i^{-1}\gamma_i\beta_i\gamma_i^{-1}) \tag{D.6}$$

Analogous results for the non-chiral partition function are

$$Z(G=2)=\sum_{m,n}\sum_{\alpha_1,\beta_1\in S_m\times S_n}Z(G=1,\alpha_1)Z(G=1,\beta_1^{-1})\\ \times\sum_{\gamma\in S_m\times S_n}\delta_{m,n}(\alpha_1^{-1}\gamma\beta_1\gamma^{-1}) \tag{D.7}$$

and

$$Z(G=2)=\sum_{m,n}\sum_{\alpha_i,\beta_i\in S_m\times S_n}Z(G=0,B=3;\alpha_1,\alpha_2,\alpha_3)Z(G=0,B=3;\beta_1^{-1},\beta_2^{-1},\beta_3^{-1})\\ \times\prod_{i=1}^{3}\sum_{\gamma_i\in S_m\times S_n}\delta_{m,n}(\alpha_i^{-1}\gamma_i\beta_i\gamma_i^{-1}) \tag{D.8}$$

References

[1] A.A. Migdal, *Recursion equations in gauge field theories*, *Sov. Phys. JETP* **42** (1975) 413 [*Zh. Eksp. Teor. Fiz.* **69** (1975) 810].

[2] D.J. Gross, *Two-dimensional QCD as a string theory*, *Nucl. Phys.* **B 400** (1993) 161 [hep-th/9212149];
D.J. Gross and W. Taylor, *Two-dimensional QCD is a string theory*, *Nucl. Phys.* **B 400** (1993) 181 [hep-th/9301068]; *Twists and Wilson loops in the string theory of two-dimensional QCD*, *Nucl. Phys.* **B 403** (1993) 395 [hep-th/9303046].

[3] V.A. Kazakov and I.K. Kostov, *Nonlinear strings in two-dimensional* $\mathrm{U}(\infty)$ *gauge theory*, *Nucl. Phys.* **B 176** (1980) 199;
V.A. Kazakov, *Wilson loop average for an arbitrary contour in two-dimensional* $\mathrm{U}(N)$ *gauge theory*, *Nucl. Phys.* **B 179** (1981) 283;
V.A. Kazakov and I.K. Kostov, *Computation of the Wilson loop functional in two-dimensional* $\mathrm{U}(\infty)$ *lattice gauge theory*, *Phys. Lett.* **B 105** (1981) 453.

[4] S. Cordes, G.W. Moore and S. Ramgoolam, *Lectures on 2D Yang-Mills theory, equivariant cohomology and topological field theories*, *Nucl. Phys.* **41** *(Proc. Suppl.)* (1995) 184 [hep-th/9411210].

[5] S. Cordes, G.W. Moore and S. Ramgoolam, *Large-N 2D Yang-Mills theory and topological string theory*, *Commun. Math. Phys.* **185** (1997) 543 [hep-th/9402107].

[6] Y. Kimura and S. Ramgoolam, *Branes, anti-branes and Brauer algebras in gauge-gravity duality*, *JHEP* **11** (2007) 078 [arXiv:0709.2158].

[7] A. Ram, *Dissertation chapter 1: representation theory*, Ph.D. thesis, http://www.math.wisc.edu/~ram/pub/dissertationChapt1.pdf, University of Californa, San Diego U.S.A. (1991).

[8] S. Ramgoolam, *Wilson loops in 2D Yang-Mills: euler characters and loop equations*, *Int. J. Mod. Phys.* **A 11** (1996) 3885 [hep-th/9412110].

[9] S.G. Naculich, H.A. Riggs and H.J. Schnitzer, *Two-dimensional Yang-Mills theories are string theories*, *Mod. Phys. Lett.* **A 8** (1993) 2223 [hep-th/9305097].

[10] S. Ramgoolam, *Comment on two-dimensional O(N) and* Sp(N) *Yang-Mills theories as string theories*, *Nucl. Phys.* **B 418** (1994) 30 [hep-th/9307085].

[11] R. Dijkgraaf, R. Gopakumar, H. Ooguri and C. Vafa, *Baby universes in string theory*, *Phys. Rev.* **D 73** (2006) 066002 [hep-th/0504221].

[12] M. Aganagic, H. Ooguri, N. Saulina and C. Vafa, *Black holes, q-deformed 2D Yang-Mills and non- perturbative topological strings*, *Nucl. Phys.* **B 715** (2005) 304 [hep-th/0411280].

[13] N. Caporaso et al., *Topological strings, two-dimensional Yang-Mills theory and Chern-Simons theory on torus bundles*, hep-th/0609129.

[14] N. Caporaso et al., *Topological strings and large-N phase transitions. II: chiral expansion of q-deformed Yang-Mills theory*, *JHEP* **01** (2006) 036 [hep-th/0511043].

[15] X. Arsiwalla, R. Boels, M. Mariño and A. Sinkovics, *Phase transitions in q-deformed 2D Yang-Mills theory and topological strings*, *Phys. Rev.* **D 73** (2006) 026005 [hep-th/0509002].

[16] S. de Haro, S. Ramgoolam and A. Torrielli, *Large-N expansion of q-deformed two-dimensional Yang-Mills theory and Hecke algebras*, *Commun. Math. Phys.* **273** (2007) 317 [hep-th/0603056].

[17] R. Dijkgraaf and G.W. Moore, *Balanced topological field theories*, *Commun. Math. Phys.* **185** (1997) 411 [hep-th/9608169].

[18] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D 70** (2004) 106007 [hep-th/0405146].

[19] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual* $N = 4$ *SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [hep-th/0111222].

[20] R. Bhattacharyya, S. Collins and R.d.M. Koch, *Exact multi-matrix correlators*, *JHEP* **03** (2008) 044 [arXiv:0801.2061].

[21] I.K. Kostov, M. Staudacher and T. Wynter, *Complex matrix models and statistics of branched coverings of 2D surfaces*, *Commun. Math. Phys.* **191** (1998) 283 [hep-th/9703189].

[22] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, *Commun. Math. Phys.* **147** (1992) 1.

[23] M.J. Crescimanno and W. Taylor, *Large-N phases of chiral QCD in two-dimensions*, *Nucl. Phys.* **B 437** (1995) 3 [hep-th/9408115].

[24] T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, math.AG/0004096.

[25] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, *Surveys Diff. Geom.* **1** (1991) 243.

[26] A. Okunkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers and matrix models I*, math.AG/0101147.

[27] R. Vakil, *The moduli space of curves and Gromov-Witten theory*, math.AG/0602347.

[28] T. Eguchi, K. Hori and S.-K. Yang, *Topological σ-models and large-N matrix integral*, *Int. J. Mod. Phys.* **A 10** (1995) 4203 [hep-th/9503017].

[29] T. Eguchi, K. Hori and C.-S. Xiong, *Gravitational quantum cohomology*, *Int. J. Mod. Phys.* **A 12** (1997) 1743 [hep-th/9605225].

JHEP06(2008)015